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# Arnold's conjecture and symplectic reduction

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**Abstract**

Fortune (1985) proved Arnold's conjecture for complex projective spaces, by exploiting the fact that $\mathbb{C}\mathbf{P}^{n-1}$ is a symplectic quotient of $\mathbb{C}^n$. In this paper, we show that Fortune's approach is universal in the sense that it is possible to translate Arnold's conjecture on any closed symplectic manifold $(Q, \Omega)$ to a critical point problem with symmetry on loops in $\mathbb{R}^{2n}$ with its standard symplectic structure.

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## 1. Introduction

Let $(Q, \Omega)$ be a closed symplectic manifold. A symplectic diffeomorphism $\varphi$ of $(Q, \Omega)$ is said to be *exact* if it can be obtained by integrating a time-dependent hamiltonian vector field. More precisely, $\varphi$ will be exact if there exists a smooth time-dependent hamiltonian $h : Q \times [0, 1] \to \mathbb{R}$ such that, defining $\varphi_t$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = X_{h_t} \circ \varphi_t, \quad \varphi_0 = \mathrm{id}_Q, \tag{1}$$

where $X_{h_t}$ satisfies $\mathrm{i}(X_{h_t})\Omega = -\mathrm{d}h_t$, one has $\varphi = \varphi_1$.

In the seventies, Arnold [1,2] conjectured that any exact symplectomorphism $\varphi_1$ of a closed symplectic manifold $(Q, \Omega)$ must have at least as many fixed points as the minimal number of critical points for a smooth real-valued function on $Q$. Moreover, if all fixed points are nondegenerate, then the lower bound is given by the minimal number of critical points for a Morse function on $Q$.

The conjecture has been proved for several classes of manifolds, by using different methods and techniques. (A more complete overview of the literature can be found in [3].)


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The most general results come from Floer's work [3]. Floer proved Arnold's conjecture for manifolds $(Q, \Omega)$ for which the class $[\Omega] \in H^2(Q, \mathbb{R})$ vanishes on $\pi_2(Q)$, and the nondegenerate part of the conjecture for monotone manifolds, i.e., manifolds for which $[\Omega]$ is positively proportional to the first Chern class $c_1(Q)$ on $\pi_2(Q)$. A crucial ingredient in Floer's proof is the existence of an action functional or, failing that, of a well-defined vector field on the space of loops whose critical points correspond to closed orbits of the hamiltonian system on $Q$. Floer's techniques have been recently refined (see e.g. [9,13,8]) to extend his results to a wider range of manifolds.

A quite different approach is due to Fortune and Weinstein [5,4]. In the case of $\mathbb{C}\mathbf{P}^{n-1}$, the action functional is multiple valued. This difficulty disappears when one considers the hamiltonian system on $\mathbb{C}\mathbf{P}^{n-1}$ as the reduction, in the sense of Marsden and Weinstein, of a hamiltonian system on $\mathbb{C}^n$. The problem can be reduced to that of finding certain families of critical orbits of the restriction of an $S^1$-invariant action functional, defined in the loop space of $\mathbb{C}^n$, to a given invariant submanifold.

The same approach has been recently used by Oh [12] and Givental [6] to get estimates for the minimal number of fixed points of exact symplectomorphisms of $\mathbb{T}^{2n} \times \mathbb{C}\mathbf{P}^k$ and toric symplectic manifolds (i.e., symplectic quotients $\mathbb{C}^n // \mathbb{T}^k$ with respect to certain linear torus actions), respectively.

In this paper, we will show how, making use of a suitable inverse reduction, Arnold's conjecture on any closed symplectic manifold $(Q, \Omega)$ can be formulated as a critical point problem with symmetry on loops in a canonical cotangent bundle $(T^*P, \mathrm{d}\theta_P)$ or, even more, as a critical point problem with symmetry on loops in some $\mathbb{R}^{2n}$ with its standard symplectic structure. The groups involved here are $\mathbb{T}^k$, for the problem in $(T^*P, \mathrm{d}\theta_P)$, and a product $\mathbb{T}^k \times \mathbb{R}^l$, for the problem in $\mathbb{R}^{2n}$. We will provide a detailed proof of these facts as well as a complete discussion of the resulting variational problems, extending previous results announced in [10].

The proof relies on the fact, showed in [7], that every symplectic manifold $(Q, \Omega)$, with $\Omega$ of finite integral rank, can be realized as a symplectic reduction (although not always as a Marsden–Weinstein reduction) of some $\mathbb{R}^{2n}$ with its standard symplectic structure.

In Section 2 we will extend the use of inverse reduction in [4] to the more general context of the Marsden–Weinstein reduction of a symplectic manifold $(M, \omega)$ with respect to the action of a connected abelian Lie group $G$. We will see how to express the fixed point problem on the reduced manifold as a fixed point problem with symmetry on, the usually simpler one, $(M, \omega)$. Then, in Section 1, we will combine these results with Gotay and Tuynman's theorem and, with an appropriate lifting of the hamiltonian system on $(Q, \Omega)$, we will prove our main result (cf. Theorems 2 and 3).

## 2. Fixed points and inverse reduction

Let $G$ be a Lie group acting symplectically on a symplectic manifold $(M, \omega)$. Let us assume that the action admits an Ad*-equivariant momentum map $J : M \to \mathcal{G}^*$, where $\mathcal{G}^*$ denotes the dual of the Lie algebra $\mathcal{G}$ of $G$.

If $\mu \in \mathcal{G}^*$ and $J^{-1}(\mu)$ is a submanifold of $M$, then there is an induced action of the stabilizer group of $\mu$, $G_\mu$, on $J^{-1}(\mu)$. We will assume that the quotient space $M_\mu = J^{-1}(\mu)/G_\mu$ is a smooth manifold and the projection $\pi_\mu : J^{-1}(\mu) \to M_\mu$ is a smooth submersion. Under these hypotheses, there is an induced symplectic form $\omega_\mu$ on $M_\mu$ and $(M_\mu, \omega_\mu)$ is known as the Marsden–Weinstein reduction of $(M, \omega)$ relative to the group action.

In what follows, we will restrict ourselves to the action $\Phi : G \times M \to M$ of a finite-dimensional connected abelian Lie group $G$ on $M$. (Notice that $G$ must be isomorphic to a product $\mathbb{T}^k \times \mathbb{R}^l$.)

Let us consider a given time-dependent hamiltonian $H_t$ on $M_\mu$ with associated time-dependent hamiltonian vector field $X_{H_t}$. We are looking for closed integral curves of $X_{H_t}$, i.e., closed solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = X_{H_t}(u(t)), \qquad u(0) = m \tag{2}$$

for any $m \in M_\mu$.

Now assume that there exists a time-dependent hamiltonian $\tilde{H} : M \times [0, 1] \to \mathbb{R}$ on $M$ such that each $\tilde{H}_t$ is a $G$-invariant extension to $M$ of the pull-back $\pi_\mu^* H_t$ and let $X_{\tilde{H}_t}$ be the corresponding hamiltonian vector field. It is easily seen that $X_{\tilde{H}_t}$ is tangent to $J^{-1}(\mu)$ and it projects on $X_{H_t}$.

Consider any $x \in J^{-1}(\mu)$ and let $\tilde{\sigma}_x$ be the solution of the initial value problem in $M$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = X_{\tilde{H}_t}(u(t)), \qquad u(0) = x. \tag{3}$$

Then $\tilde{\sigma}_x$ lies in $J^{-1}(\mu)$ and $\sigma_m = \pi_\mu \circ \tilde{\sigma}_x$ is the solution of (2) with $m = \pi_\mu(x)$. For each $g \in G$, $\Phi_g \circ \tilde{\sigma}_x = \tilde{\sigma}_{\Phi_g(x)}$ will also be an integral curve of $X_{\tilde{H}_t}$ projecting on $\sigma_m$. In fact, there is a one-to-one correspondence between integral curves $\sigma_m$ of $X_{H_t}$ and families of integral curves of $X_{\tilde{H}_t}$ with initial values at the points of the orbit $\pi_\mu^{-1}(m)$.

The curve $\sigma_m$ will be closed if and only if each $\tilde{\sigma}_x$ in the corresponding family satisfies $\tilde{\sigma}_x(0) = \Phi_{g_0}(\tilde{\sigma}_x(1))$ for certain $g_0 \in G$. Since $G$ is a connected abelian group, the exponential mapping $\exp : \mathcal{G} \to G$ is onto. If we pick any $\xi \in \exp^{-1}(g_0)$ and define $\tilde{\sigma}_x^\xi(t) = \Phi_{g_\xi(t)}(\tilde{\sigma}_x(t))$, where $g_\xi$ denotes the curve $t \mapsto \exp(t\xi)$ in $G$, then $\tilde{\sigma}_x^\xi$ is a closed integral curve of $X_{\tilde{H}_t} + \xi_M$, with $\xi_M$ being the infinitesimal generator of the action on $M$ corresponding to $\xi \in \mathcal{G}$, as we next prove.

It is obvious that $\tilde{\sigma}_x^\xi(0) = \tilde{\sigma}_x^\xi(1) = x$.

Now, differentiation of the expression of $\tilde{\sigma}_x^\xi(t)$ with respect to $t$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\sigma}_x^\xi(t) = T_{\tilde{\sigma}_x(t)}\Phi_{g_\xi(t)}(X_{\tilde{H}_t}(\tilde{\sigma}_x(t))) + T_{g_\xi(t)}\Phi_{\tilde{\sigma}_x(t)}(\dot{g}_\xi(t)). \tag{4}$$

Since $X_{\tilde{H}_t}$ is $G$-invariant, the first term turns out to be $X_{\tilde{H}_t}(\tilde{\sigma}_x^\xi(t))$.

On the other hand, denoting by $L$ left translation in $G$ and by $e$ the unit element of $G$, we have

$$\begin{aligned} T_{g_\xi(t)}\Phi_{\tilde{\sigma}_x(t)}(\dot{g}_\xi(t)) &= T_e(\Phi_{\tilde{\sigma}_x(t)} \circ L_{g_\xi(t)})(\xi) = T_e(\Phi_{g_\xi(t)} \circ \Phi_{\tilde{\sigma}_x(t)})(\xi) \\ &= T_{\tilde{\sigma}_x(t)}\Phi_{g_\xi(t)}(\xi_M(\tilde{\sigma}_x(t))) = \xi_M(\tilde{\sigma}_x^\xi(t)) \end{aligned} \tag{5}$$

and the desired result follows.

Thus, if $J_\xi = \langle J, \xi \rangle$ is the hamiltonian associated to $\xi_M$, then $\tilde{\sigma}_x^\xi$ is the solution of the initial value problem in $M$:

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = X_{\tilde{H}_t + J_\xi}(u(t)), \qquad u(0) = x \tag{6}$$

and it is closed.

Notice that $J_\xi$ is also $G$-invariant and the reduced vector field of $X_{\tilde{H}_t + J_\xi}$ is again $X_{H_t}$.

For a nonfree action [1] the element $g_0$ mentioned above is not unique. It can be replaced by any $g \in g_0 G_x$, where $G_x$ denotes the stabilizer group of $x$. Moreover, the correspondence $\xi \mapsto \tilde{\sigma}_x^\xi$ is not one-to-one. More explicitly, since the curves $\tilde{\sigma}_x^\xi$ satisfy $\Phi_g \circ \tilde{\sigma}_x^\xi = \tilde{\sigma}_{\Phi_g(x)}^\xi$, the stabilizer of $\tilde{\sigma}_x^\xi(t)$ is $G_x$ for each $t$. On the other hand,

$$\tilde{\sigma}_x^\eta(t) = \Phi_{g_{\eta-\xi}(t)}(\tilde{\sigma}_x^\xi(t)) \tag{7}$$

and

$$\tilde{\sigma}_x^\eta = \tilde{\sigma}_x^\xi \Leftrightarrow \exp(t(\eta - \xi)) \in G_x, \quad \forall t \Leftrightarrow \eta - \xi \in \mathcal{G}_x, \tag{8}$$

where $\mathcal{G}_x$ denotes the Lie algebra of $G_x$, which can also be characterized as $\mathcal{G}_x = \{\xi \in \mathcal{G} \mid \xi_M(x) = 0\}$.

Now we are ready to state the following proposition.

**Proposition 1.** *To each fixed point $m$ of the exact symplectomorphism $\psi_1$ induced by the hamiltonian $H_t$ on $M_\mu$ there corresponds a family of closed curves in $J^{-1}(\mu)$ : $\mathcal{F}_{m,g_0} = \{\tilde{\sigma}_x^\xi \in C^\infty(S^1, M) \mid \tilde{\sigma}_x^\xi$ solves (6), $x \in \pi_\mu^{-1}(m), \xi \in \exp^{-1}(g_0 G_x) \bmod \mathcal{G}_x\}$, for certain $g_0 \in G$. This family is diffeomorphic to the product of an orbit $G/G_x$ and the projection of $\exp^{-1}(G_x)$ on $\mathcal{G}/\mathcal{G}_x$.*

## 3. Lifting to $\mathbb{R}^{2n}$

Let $(Q, \Omega)$ be a closed symplectic manifold and consider a time-dependent hamiltonian $h_t$ on $Q$ with associated hamiltonian vector field $X_{h_t}$. Denoting by $\varphi_t$ the flow of $X_{h_t}$, we are concerned with the number of fixed points of $\varphi_1$.

As mentioned above, our purpose is to translate the fixed point problem on $(Q, \Omega)$ to a critical point problem on loops in $\mathbb{R}^{2n}$. We have been motivated by the following theorem.

**Theorem 1** (Gotay and Tuynman [7]). *Every symplectic manifold $(Q, \Omega)$, with $\Omega$ of finite integral rank, can be realized as a reduction of some $\mathbb{R}^{2n}$ with its standard symplectic structure.*

Since $Q$ is compact, the condition of $\Omega$ having finite integral rank is automatically satisfied in our case.

[1] In Proposition 1 of [10], the action of $G$ on $J^{-1}(\mu)$ was assumed to be free. We derive here the corresponding result for a general action.

On the other hand, reduction in Theorem 1 must be understood in the following sense.

If $(M,\omega)$ is a symplectic manifold and $N$ is a submanifold such that the pull-back $\omega_N$ of $\omega$ to $N$ has constant rank and $\ker\omega_N$ is fibrating, then the quotient (symplectic) manifold $M_N = N/\ker\omega_N$ is called the reduction of $M$ by $N$.

Therefore, we cannot directly apply the results of Section 2 in order to prove our statement.

We will develop our proof in three stages, according to the scheme of proof of Theorem 1.

*Step* 1: Let us consider the cotangent bundle $\tau_Q : T^*Q \to Q$ and let $\theta_Q$ be the Liouville 1-form on $T^*Q$. The zero section $Z_Q$ is a symplectic submanifold of $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$, and it is canonically symplectomorphic to $(Q,\Omega)$. Therefore, $(Q,\Omega)$ can be realized as the reduction of $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ by $Z_Q$. This is the first step in the proof of Theorem 1.

Now, we need to lift the fixed point problem from $(Q,\Omega)$ to $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$. The next four lemmas will make the job.

**Lemma 1.** *There is a one-to-one correspondence between symplectomorphisms of* $(Q,\Omega)$ *and symplectomorphisms of* $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ *preserving* $\theta_Q$.

*Proof.* It is well known that a diffeomorphism of $T^*Q$ is the lift of a diffeomorphism of $Q$ if and only if it preserves $\theta_Q$, the latter being called *homogeneous* diffeomorphisms of $T^*Q$.

Thus, to each symplectomorphism $\varphi$ of $(Q,\Omega)$ one can associate the homogeneous diffeomorphism

$$T^*\varphi^{-1} : T^*Q \to T^*Q, \qquad (q,\beta_q) \mapsto (\varphi(q), \beta_q \circ T_{\varphi(q)}\varphi^{-1}). \tag{9}$$

To show that this diffeomorphism preserves the whole symplectic form on $T^*Q$, it is enough to check that $(T^*\varphi^{-1})^*\tau_Q^*\Omega = \tau_Q^*\Omega$. But this is clear from the property $\tau_Q \circ T^*\varphi^{-1} = \varphi \circ \tau_Q$ and the fact that $\varphi$ is symplectic.

Conversely, given a homogeneous symplectomorphism $\psi$ of $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$, there exists a unique diffeomorphism $\varphi$ of $Q$ such that $\psi = T^*\varphi^{-1}$, and this diffeomorphism is symplectic, since: $\psi^*(\mathrm{d}\theta_Q + \tau_Q^*\Omega) = \mathrm{d}\theta_Q + \tau_Q^*\Omega$ implies $\psi^*\tau_Q^*\Omega = \tau_Q^*\Omega$, that is, $\tau_Q^*(\varphi^*\Omega - \Omega) = 0$. But $\tau_Q^*$ is injective, so that $\varphi^*\Omega = \Omega$, as was to be proved. □

**Lemma 2.** *There is a one-to-one correspondence between smooth hamiltonian isotopies* $\varphi_t$ *of* $(Q,\Omega)$ *and smooth hamiltonian isotopies* $\psi_t$ *of* $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ *such that* $\psi_t$ *is homogeneous for each* $t$.

*Proof.* Let $\varphi_t$ be the flow induced by a hamiltonian $h_t$ on $(Q,\Omega)$, and define $\psi_t = T^*\varphi_t^{-1}$. Then, the $\psi_t$ constitute a smooth family of homogeneous symplectomorphisms connecting $\psi_1$ with the identity map.

Now, let us consider the vector field defined by differentiating $\psi_t$ in $t$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \tilde{X}_t \circ \psi_t. \tag{10}$$

From $\tau_Q \circ \psi_t = \varphi_t \circ \tau_Q$, it follows that

$$T\tau_Q(\tilde{X}_t) = X_{h_t} \circ \tau_Q. \tag{11}$$

Therefore, $\mathrm{i}(\tilde{X}_t)(\tau_Q^* \Omega) = -\mathrm{d}(h_t \circ \tau_Q)$.

On the other hand,

$$\mathrm{i}(\tilde{X}_t)\, \mathrm{d}\theta_Q = \mathcal{L}_{\tilde{X}_t}\theta_Q - \mathrm{d}(\theta_Q(\tilde{X}_t)), \tag{12}$$

where $\mathcal{L}$ stands for the Lie derivative. Since the $\psi_t$ are homogeneous, $\mathcal{L}_{\tilde{X}_t}\theta_Q = 0$ and, finally,

$$\mathrm{i}(\tilde{X}_t)(\mathrm{d}\theta_Q + \tau_Q^*\Omega) = -\mathrm{d}(\theta_Q(\tilde{X}_t) + h_t \circ \tau_Q). \tag{13}$$

Thus, the family $\psi_t$ is generated by the hamiltonian

$$H_t = h_t \circ \tau_Q + \theta_Q(\tilde{X}_t). \tag{14}$$

In terms of the original hamiltonian $h_t$

$$H_t(q, \beta_q) = h_t(q) + \beta_q(X_{h_t}(q)), \quad \forall (q, \beta_q) \in T^*Q. \tag{15}$$

Conversely, let $\psi_t$ be the flow induced by a hamiltonian $H_t$ on $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ and suppose that, for each $t$, $\psi_t$ is homogeneous. Then, there exists a family $\varphi_t$ of symplectomorphisms of $(Q, \Omega)$ such that $\psi_t = T^*\varphi_t^{-1}$, $\forall t$.

If $j : Z_Q \hookrightarrow T^*Q$ denotes the inclusion of the zero section, then $H_t \circ j$ is a hamiltonian on the symplectic submanifold $Z_Q$, which in turn defines a hamiltonian $h_t$ on $Q$ by

$$H_t \circ j = h_t \circ \tau_Q \circ j. \tag{16}$$

A straightforward calculation shows that this hamiltonian generates the family $\varphi_t$. □

The fixed points $(q, \beta_q)$ of $\psi_1$ are characterized by the two conditions $\varphi_1(q) = q$ and $\beta_q \circ T_q\varphi_1^{-1} = \beta_q$. The last one is equivalent to the existence of an eigenvector of $T_q\varphi_1$ with eigenvalue equal to 1, namely, the vector defined by $\mathrm{i}(v_q)\Omega_q = \beta_q$. Hence, the following lemma holds.

**Lemma 3.** *To each fixed point $q$ of $\varphi_1$, one can univocally associate a subspace $V_q \subset T_q^*Q$ of fixed points of $\psi_1$, whose dimension is equal to the multiplicity of $q$: $m(q) = \dim \ker(T_q\varphi_1 - \mathrm{id}_{T_qQ})$.*

Moreover, for the nondegenerate case, we have the following lemma.

**Lemma 4.** *$\psi_1$ is nondegenerate if and only if $\varphi_1$ is nondegenerate.*

*Under these circumstances, there is a one-to-one correspondence between fixed points of both symplectomorphisms.*

*Proof.* Suppose that $\varphi_1$ is nondegenerate and let $(q, \beta_q)$ be a fixed point of $\psi_1$. By the preceding remarks, it must be $\beta_q = 0_q$. Let $w_{(q, 0_q)} \in T_{(q, 0_q)}T^*Q$ be an eigenvector of $T_{(q, 0_q)}\psi_1$ with eigenvalue equal to 1. The nondegeneracy of $q$ implies $w_{(q, 0_q)} \in \ker T_{(q, 0_q)}\tau_Q$. As a consequence, the 1-form $\mathrm{i}(w_{(q, 0_q)})\, \mathrm{d}_{(q, 0_q)}\theta_Q + (\tau_Q^*\Omega)_{(q, 0_q)})$ is a 1-form on $T_{(q, 0_q)}Z_Q$.

Using the canonical identification between $Z_Q$ and $Q$, it is not difficult to check that this 1-form defines a $T_q\varphi_1$-invariant 1-form on $T_qQ$. Since the last 1-form must be zero, we conclude that $w_{(q,0_q)} = 0$.

Conversely, suppose that $\psi_1$ is nondegenerate and consider a fixed point $q$ of $\varphi_1$. Then, $(q, 0_q)$ is a nondegenerate fixed point of $\psi_1$. Using again the identification between $Z_Q$ and $Q$, it is easily seen that to each vector $v_q \in T_qQ$ one can univocally associate a vector $w_{(q,0_q)} \in T_{(q,0_q)}T^*Q$, tangent to $Z_Q$, which must be $T_{(q,0_q)}\psi_1$-invariant if $v_q$ is $T_q\varphi_1$-invariant. Since $(q, 0_q)$ is nondegenerate, $w_{(q,0_q)}$, and hence $v_q$, must be zero. □

The fixed point problem in $(Q, \Omega)$ can be lifted in this way to an equivalent fixed point problem in $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$. We must search for the fixed points of the exact symplectomorphism $\psi_1$, induced by a hamiltonian $H_t$ of the form (15) on $T^*Q$. In the case of general exact symplectomorphisms, we must count, instead of single fixed points, whole subspaces of fixed points of $\psi_1$.

*Step* 2: The symplectic manifold $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ is obtained as a reduction, in the sense of Marsden and Weinstein, of a canonical cotangent bundle $(T^*P, \mathrm{d}\theta_P)$ as follows (see [7]).

Since $\Omega$ has finite integral rank, $\Omega = \mu_1 c_1 + \cdots + \mu_k c_k$, where $c_1, \ldots, c_k$ are integral closed 2-forms and $\mu_1, \ldots, \mu_k$ are real constants. According to the prequantization procedure of Kostant and Souriau, for each $i = 1, \ldots, k$ there exists a principal $S^1$-bundle $\pi_i : P_i \to Q$, with a connection $A_i$ whose curvature satisfies $\mathrm{d}A_i = \pi_i^* c_i$. The fiber product of the $P_i$, $\pi : P \to Q$, is a principal $\mathbb{T}^k$-bundle with connection $A = (A_1, \ldots, A_k)$ and curvature $\mathrm{d}A = (\pi^* c_1, \ldots, \pi^* c_k)$.

Let us denote by $\Phi$ the action of $\mathbb{T}^k$ on $P$, and by $\Phi^{T^*}$ the cotangent lifting of $\Phi$ to $(T^*P, \mathrm{d}\theta_P)$. There is a canonically defined momentum map $J : T^*P \to \mathbb{R}^k$ given by

$$J(p, \gamma_p) = (T_e\Phi_p)^*\gamma_p = \gamma_p \circ T_e\Phi_p, \tag{17}$$

where $e = (1, \ldots, 1)$ is the unit element of $\mathbb{T}^k$.

The cotangent bundle reduction theorem (see [11]) provides a symplectic diffeomorphism $\nu_{A,\mu}$ between the symplectic manifolds $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*(\mu_1 c_1 + \cdots + \mu_k c_k))$ and $\left(J^{-1}(\mu)/\mathbb{T}^k, (\mathrm{d}\theta_P)_\mu\right)$, where $\mu = (\mu_1, \ldots, \mu_k)$.

Our purpose is to find an invariant extension $\hat{H}_t$ of $H_t \circ \nu_{A,\mu} \circ \pi_\mu$ to $T^*P$, where $H_t$ is the hamiltonian (15).

The existence of such an extension is guaranteed by the fact that $\mathbb{T}^k$ is compact. It should also be clear from Section 2 that any invariant extension allows us to lift the problem to $T^*P$. However, we will next describe a natural construction of $\hat{H}_t$ in terms of the connection $A$.

Given the connection $A$ on $P$, one can construct a lift $\pi_A : T^*P \to T^*Q$ of $\pi$ to the cotangent bundles, as explained in [14].

For each $p \in P$ we have an exact sequence

$$0 \longrightarrow \mathbb{R}^k \xrightarrow{T_e\Phi_p} T_pP \xrightarrow{T_p\pi} T_{\pi(p)}Q \longrightarrow 0, \tag{18}$$

The connection $A$ gives a splitting of (18), which in turn defines a linear map $\Gamma_p : T_{\pi(p)}Q \to T_pP$ (the image of $\Gamma_p$ consisting of all horizontal vectors in $T_pP$).

We also have a splitting of the dual exact sequence

$$0 \longleftarrow \mathbb{R}^k \overset{(T_e\Phi_p)^*}{\longleftarrow} T_p^*P \overset{(T_p\pi)^*}{\longleftarrow} T^*_{\pi(p)}Q \longleftarrow 0 \tag{19}$$

and $\pi_A$ is defined fiberwise by

$$\pi_A \mid_{T_p^*P} = \Gamma_p^*. \tag{20}$$

The map $\pi_A$ is a lift of $\pi$ in the sense that it makes the diagram

$$\begin{array}{ccc} T^*P & \overset{\pi_A}{\longrightarrow} & T^*Q \\ \tau_P \downarrow & & \downarrow \tau_Q \\ P & \overset{\pi}{\longrightarrow} & Q \end{array} \tag{21}$$

commutative. Moreover, $\pi_A$ is constant along the orbits of the lifted action $\Phi^{T^*}$.

From the explicit construction of $\nu_{A,\mu}$ in [11], it is easily seen that the diagram

$$\begin{array}{ccc} J^{-1}(\mu) & \overset{\pi_\mu}{\longrightarrow} & J^{-1}(\mu)/\mathbb{T}^k \\ i_\mu \downarrow & & \downarrow \nu_{A,\mu} \\ T^*P & \overset{\pi_A}{\longrightarrow} & T^*Q \end{array} \tag{22}$$

where $i_\mu$ is the inclusion of $J^{-1}(\mu)$, also commutes.

From the commutativity of this diagram, one can immediately derive the following lemma.

**Lemma 5.** *The time-dependent hamiltonian $\hat{H}_t = H_t \circ \pi_A$ is a $\mathbb{T}^k$-invariant extension of $H_t \circ \nu_{A,\mu} \circ \pi_\mu$ to $T^*P$.*

In terms of the hamiltonian $h_t$ on $Q$, we have

$$\hat{H}_t(p, \gamma_p) = h_t(\pi(p)) + \gamma_p(\Gamma_p(X_{h_t}(\pi(p)))), \quad \forall (p, \gamma_p) \in T^*P. \tag{23}$$

On the other hand, if $V_q$ is the subspace of Lemma 3, then the points in $J^{-1}(\mu)$ projecting onto $V_q$ constitute the $\mathbb{T}^k$-orbits of points in $\langle \mu, A_p \rangle + (T_p\pi)^*(V_q) \subset T_p^*P$, for any $p \in \pi^{-1}(q)$. Indeed, if $(p, \gamma_p)$ belongs to $J^{-1}(\mu)$, then $\gamma_p$ must be of the form

$$\gamma_p = \langle \mu, A_p \rangle + \beta_{\pi(p)} \circ T_p\pi, \tag{24}$$

where $\beta_{\pi(p)} \in T^*_{\pi(p)}Q$. Besides,

$$(\nu_{A,\mu} \circ \pi_\mu)(p, \gamma_p) = \pi_A(p, \gamma_p) = (\pi(p), \gamma_p \circ \Gamma_p) = (\pi(p), \beta_{\pi(p)}). \tag{25}$$

Thus,

$$(\nu_{A,\mu} \circ \pi_\mu)^{-1}(V_q) = \bigcup_{p \in \pi^{-1}(q)} (\langle \mu, A_p \rangle + (T_p\pi)^*(V_q)). \tag{26}$$

The invariance of $A$ and $T\pi$ imply that this union is in fact a union of orbits of the action $\Phi^{T^*}$ on $T^*P$.

From the definition of $\hat{H}_t$, it is clear that the integral curves of the associated vector field $X_{\hat{H}_t}$ project on horizontal lifts to $P$ of integral curves of $X_{h_t}$.

Now, let $\hat{\sigma}_{(p,\gamma_p)}$ and $\hat{\sigma}_{(p,\gamma'_p)}$ be two integral curves of $X_{\hat{H}_t}$ with (different) initial values in $\langle \mu, A_p\rangle + (T_p\pi)^*(V_q)$ and corresponding to the same fixed point $q$ in $Q$. Let $g_0$ and $g'_0$ be the elements of $\mathbb{T}^k$ relating their initial and final values. Then it is easy to check that since both $\tau_P \circ \hat{\sigma}_{(p,\gamma_p)}$ and $\tau_P \circ \hat{\sigma}_{(p,\gamma'_p)}$ are equal to the horizontal lift to $P$ of the curve $\varphi_t(q)$, $g_0$ and $g'_0$ must coincide.

Combining the results of Steps 1 and 2, and using Proposition 1 particularized to the case of a free torus action, we can state the following theorem.

**Theorem 2.** *Let us consider the initial value problem in* $T^*P$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = X_{\hat{H}_t + J_\lambda}(u(t)), \qquad u(0) = (p, \gamma_p) \tag{27}$$

*with* $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. *Then, to each fixed point* $q$ *of* $\varphi_1$ *there corresponds a family of closed curves in* $J^{-1}(\mu) \subset T^*P$: $\mathcal{F}_{q,a} = \{\hat{\sigma}^\lambda_{(p,\gamma_p)} \in C^\infty(S^1, T^*P) \mid \hat{\sigma}^\lambda_{(p,\gamma_p)}$ *solves* (27), $\lambda_i = a_i \,(\mathrm{mod}\, 2\pi), \forall i, \pi(p) = q, \gamma_p \in \langle \mu, A_p\rangle + (T_p\pi)^*(V_q)\}$, *for certain* $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$. *This family is diffeomorphic to* $V_q \times \mathbb{T}^k \times \mathbb{Z}^k$.

By Hamilton's principle, the closed integral curves of $X_{\hat{H}_t + J_\lambda}$ are exactly the critical points of the action functional

$$S_{\hat{H},\lambda}(u) = \int_u \theta_P - \int_0^1 \left(\hat{H}_t(u(t)) + J_\lambda(u(t))\right) \mathrm{d}t \tag{28}$$

defined on closed loops $u$ in $T^*P$.

Notice that this functional is $\mathbb{T}^k$-invariant, where the (free) action of $\mathbb{T}^k$ on the loop space of $T^*P$ is the obvious one: $(\Phi_g(u))(t) = \Phi_g(u(t))$.

We are looking for critical loops satisfying

$$J(u(t)) = \mu, \quad \forall t \in S^1. \tag{29}$$

Since the hamiltonians considered are $\mathbb{T}^k$-invariant, this pointwise condition is equivalent on critical loops to the constraint $\mathcal{J}(u) = \mu$, where

$$\mathcal{J}(u) = \int_0^1 J(u(t))\, \mathrm{d}t \tag{30}$$

takes its values in $\mathbb{R}^k$. This map $\mathcal{J}$ can be seen as a momentum mapping for the action of $\mathbb{T}^k$ on the loop space of $T^*P$.

Following [4], one can apply now Lagrange's multipliers theorem and identify the critical points above with the critical points of the restriction $f$ of $S_{\hat{H},0}$ to $\mathcal{J}^{-1}(\mu)$. Notice that because the action of $\mathbb{T}^k$ is free, the map $\mathcal{J}$ will be a submersion at every $u \in \mathcal{J}^{-1}(\mu)$.

Let us compute now the difference between critical values of $f$ corresponding to the same fixed point in $Q$.

If $u$ is a critical loop of $S_{\hat{H},\lambda}$ belonging to the family associated to a closed curve $\varphi_t(q)$ in $Q$, then:

$$
\begin{aligned}
f(u) &= \int_0^1 (\theta_P)_{u(t)}(\dot{u}(t))\,\mathrm{d}t - \int_0^1 \hat{H}_t(u(t))\,\mathrm{d}t \\
&= \int_0^1 (\theta_P)_{u(t)}(\lambda_{T^*P}(u(t)))\,\mathrm{d}t + \int_0^1 u(t)(T_{u(t)}\tau_P(X_{\hat{H}_t}(u(t))))\,\mathrm{d}t \\
&\quad - \int_0^1 \hat{H}_t(u(t))\,\mathrm{d}t = \int_0^1 J_\lambda(u(t))\,\mathrm{d}t - \int_0^1 h_t((\pi \circ \tau_p \circ u)(t))\,\mathrm{d}t \\
&= \langle \mu, \lambda\rangle - \int_0^1 h_t(\varphi_t(q))\,\mathrm{d}t
\end{aligned}
\tag{31}
$$

and the difference between any two critical values of $f$ corresponding to a fixed point $q \in Q$ is of the form $\langle \mu, \eta\rangle$ with $\eta \in \exp^{-1}(e) = \mathbb{Z}^k$.

Thus, to each fixed point of the hamiltonian system on $Q$ there corresponds a tower of critical values of $f$, parametrized by the set $\{\langle \mu, \eta\rangle \mid \eta \in \mathbb{Z}^k\}$.

*Step* 3: First of all, we will briefly recall the last step in Gotay and Tuynman's proof.

They first take a closed equivariant embedding $P \hookrightarrow \mathbb{R}^n$ in some orthogonal representation space for $\mathbb{T}^k$. (The existence of such an embedding is a well-known theorem of Mostow and Palais.)

In order to obtain $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ as a Marsden–Weinstein reduction of the corresponding $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$, they first show that the ideal in $C^\infty(\mathbb{R}^n)$ of all functions vanishing on $P$ is generated over $C^\infty(\mathbb{R}^n)$ by a finite collection $\{f_1, \ldots, f_l\}$ of $\mathbb{T}^k$-invariant functions.

Then, they define, for each $i = 1, \ldots, l$, a hamiltonian $F_i = f_i \circ \tau_n$, where $\tau_n : \mathbb{R}^{2n} \to \mathbb{R}^n$ is the canonical cotangent projection. The flows of the corresponding hamiltonian vector fields define a symplectic $\mathbb{R}^l$-action with momentum map $F = (F_1, \ldots, F_l)$. The Marsden–Weinstein reduction of $\mathbb{R}^{2n}$ relative to this action, $F^{-1}(\mathbf{0})/\mathbb{R}^l$, is symplectomorphic to $(T^*P, \mathrm{d}\theta_P)$.

Since the functions $f_i$ have been chosen $\mathbb{T}^k$-invariant, the $\mathbb{R}^l$-action commutes with the cotangent lifting of the $\mathbb{T}^k$-action on $\mathbb{R}^n$, and there is a well-defined symplectic $(\mathbb{T}^k \times \mathbb{R}^l)$-action with momentum map $K \times F$, where $K$ is the canonical momentum map for the torus action. Now, it follows that $(T^*Q, \mathrm{d}\theta_Q + \tau_Q^*\Omega)$ is symplectomorphic to the reduced manifold $(K \times F)^{-1}(\mu; \mathbf{0})/(\mathbb{T}^k \times \mathbb{R}^l)$.

Notice that the $\mathbb{R}^l$-action is simply given by

$$
\Psi_{(b_1,\ldots,b_l)}(x, y) = \left(x, y - \sum_{i=1}^{l} b_i d_x f_i\right), \quad \forall (x, y) \in \mathbb{R}^{2n} \tag{32}
$$

and $F^{-1}(\mathbf{0}) = T_P^*\mathbb{R}^n$. The projection $\pi_P : T_P^*\mathbb{R}^n \to T^*P$ is just the projection associated to the direct sum decomposition

$$T_x^*\mathbb{R}^n = T_x^*P \oplus \mathrm{span}\{d_x f_1, \ldots, d_x f_l\} \tag{33}$$

for each $x \in P$, and it is $\mathbb{T}^k$-equivariant. The orbits of the $\mathbb{R}^l$-action on $T_P^*\mathbb{R}^n$ are diffeomorphic to $\mathbb{R}^{\mathrm{codim}\, P}$.

On the other hand, $\pi_P(K^{-1}(\mu) \cap F^{-1}(\mathbf{0})) = J^{-1}(\mu)$ and the projection $(K \times F)^{-1}(\mu; \mathbf{0}) \to (K \times F)^{-1}(\mu; \mathbf{0})/(\mathbb{T}^k \times \mathbb{R}^l)$ is $\pi_\mu \circ \pi_P|_{K^{-1}(\mu)\cap F^{-1}(\mathbf{0})}$.

Next we prove the following lemma.

**Lemma 6.** *There exists an $(\mathbb{T}^k \times \mathbb{R}^l)$-invariant extension $\tilde{H}_t$ of $H_t \circ \nu_{A,\mu} \circ \pi_\mu \circ \pi_P|_{K^{-1}(\mu)\cap F^{-1}(\mathbf{0})}$ to $\mathbb{R}^{2n}$.*

*Proof.* First notice that it will suffice to construct an $\mathbb{R}^l$-invariant extension $\tilde{H}_t$ of $\hat{H}_t \circ \pi_P$ which is also $\mathbb{T}^k$-invariant.

Since $\mathbb{T}^k$ is compact, we can always find a $\mathbb{T}^k$-invariant tubular neighborhood $U$ of $P$ in $\mathbb{R}^n$, a smooth $\mathbb{T}^k$-equivariant retraction $r : U \to P$, and a $\mathbb{T}^k$-invariant partition of unity $\{\rho_0, \rho_1\}$ subordinate to the open cover $\{U, \mathbb{R}^n - P\}$ of $\mathbb{R}^n$.

Now define

$$\tilde{H} : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}, \qquad ((x,y),t) \mapsto \rho_0(x)\left(\hat{H}_t \circ \pi_P\right)(r(x), y). \tag{34}$$

It is easily seen that this hamiltonian gives an $\mathbb{R}^l$-invariant extension $\tilde{H}_t$ of $\hat{H}_t \circ \pi_P$, which is also $\mathbb{T}^k$-invariant. □

As in Step 2, it is not hard to verify that the points in $\mathbb{R}^{2n}$ projecting onto $V_q$ constitute the $(\mathbb{T}^k \times \mathbb{R}^l)$-orbits of points belonging to $\langle \mu, A_x \rangle + (T_x\pi)^*(V_q)$, for any $x \in \pi^{-1}(q)$.

Now, we can state the following theorem.

**Theorem 3.** *Let us consider the initial value problem in $\mathbb{R}^{2n}$:*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = X_{\tilde{H}_t + (K\times F)_\xi}(u(t)), \qquad u(0) = (x, y) \tag{35}$$

*with $\xi = (\lambda_1, \ldots, \lambda_k; b_1, \ldots, b_l) \in \mathbb{R}^k \times \mathbb{R}^l$. Then, to each fixed point $q$ of $\varphi_1$ there corresponds a family of closed curves in $(K \times F)^{-1}(\mu, \mathbf{0}) \subset \mathbb{R}^{2n}$: $\tilde{\mathcal{F}}_{q,a} = \{\tilde{\sigma}^\xi_{(x,y)} \in C^\infty(S^1, \mathbb{R}^{2n}) \mid \tilde{\sigma}^\xi_{(x,y)}$ solves (35), $\lambda_i = a_i (\mathrm{mod}\, 2\pi)\, \forall i$, $\pi(x) = q, y \in \langle \mu, A_x \rangle + (T_x\pi)^*(V_q) + \mathrm{span}\{d_x f_1, \ldots, d_x f_l\}\}$, for certain $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$. This family is diffeomorphic to $V_q \times \mathbb{T}^k \times \mathbb{R}^{n-k-\dim Q} \times \mathbb{Z}^k$.*

**Remark.** Notice that a closed curve $\tilde{\sigma}^\xi_{(x,y)}$ is determined by the data $(x, y)$ and $\lambda$.

As before, the fixed point problem in $Q$ can be translated to the problem of finding the critical points of the family of action functionals in the loop space of $\mathbb{R}^{2n}$:

$$
\begin{aligned}
S_{\tilde{H},\xi}(u) &= \int_u \theta_0 - \int_0^1 \left(\tilde{H}_t(u(t)) + (K \times F)_\xi(u(t))\right) \mathrm{d}t \\
&= \int_u \theta_0 - \int_0^1 \tilde{H}_t(u(t))\,\mathrm{d}t - \int_0^1 K_\lambda(u(t))\,\mathrm{d}t - \int_0^1 F_b(u(t))\,\mathrm{d}t
\end{aligned} \tag{36}
$$

with $\xi = (\lambda; b) \in \mathbb{R}^k \times \mathbb{R}^l$, subject to the constraints:

$$
\mathcal{K}(u) = \int_0^1 K(u(t))\,\mathrm{d}t = \mu, \qquad \mathcal{F}(u) = \int_0^1 F(u(t))\,\mathrm{d}t = 0. \tag{37}
$$

The action of $\mathbb{T}^k \times \mathbb{R}^l$ is not free now and $\mathcal{K}^{-1}(\mu) \cap \mathcal{F}^{-1}(\mathbf{0})$ will not be in general a submanifold. Nevertheless, one may still consider the components of $\lambda$ as Lagrange's multipliers, as we next explain.

**Proposition 2.** *With the previous notation, the set $\mathcal{K}^{-1}(\mu)$ is a submanifold of codimension $k$ in the space of free loops on $\mathbb{R}^{2n}$.*

*Proof.* Let us consider a loop $u \in \mathcal{K}^{-1}(\mu)$. The tangent map $T_u\mathcal{K}$ fails to be surjective if and only if the stabilizer group of $u$ in $\mathbb{T}^k$ is not discrete, i.e. if and only if the corresponding Lie subalgebra $\{\lambda \in \mathbb{R}^k \mid \lambda_{\mathbb{R}^{2n}} \circ u = 0\}$ is not zero ($\lambda_{\mathbb{R}^{2n}}$ denotes, as is customary in this paper, the vector field associated to the element $\lambda$ in the Lie algebra of $\mathbb{T}^k$ by means of its action on $\mathbb{R}^{2n}$).

If the stabilizer of $u$ is not discrete, it must contain a subgroup isomorphic to $S^1$. Hence, the stabilizer algebra must contain an element $\lambda \in (2\pi\mathbb{Z})^k$ which generates the circle. This $\lambda$ will belong to the kernel of $\mu$, because

$$
\lambda_{\mathbb{R}^{2n}} \circ u = 0 \Rightarrow K_\lambda \circ u = 0 \Rightarrow \mathcal{K}_\lambda(u) = \langle \mu, \lambda \rangle = 0. \tag{38}
$$

But $\mu = (\mu_1, \ldots, \mu_k)$ comes from the decomposition $\Omega = \mu_1 c_1 + \cdots + \mu_k c_k$, so that we may assume that $\mu_1, \ldots, \mu_k$ are independent over $\mathbb{Z}$, hence $\ker \mu$ does not contain any nonzero $\lambda \in (2\pi\mathbb{Z})^k$.

Therefore, $\mathcal{K}$ is a submersion at every $u \in \mathcal{K}^{-1}(\mu)$ and $\mathcal{K}^{-1}(\mu)$ is a submanifold. □

Thus, the critical points of the family (36) satisfying (37) are exactly the critical points of the family of functionals $f_b$, $b \in \mathbb{R}^l$ defined by the restriction of the functionals

$$
S_{\tilde{H},(0,b)} = \int_u \theta_0 - \int_0^1 \tilde{H}_t(u(t))\,\mathrm{d}t - \int_0^1 F_b(u(t))\,\mathrm{d}t, \tag{39}
$$

to the submanifold $\mathcal{K}^{-1}(\mu)$, and satisfying $\mathcal{F}(u) = 0$.

It must be noticed that the role played by the parameters $b$ is quite different to the role played by the Lagrange multipliers $\lambda$ because they do not produce any splitting of the critical subsets. To each critical $\mathbb{T}^k$-orbit of the functional $f$ in Step 2 there corresponds a critical $(\mathbb{T}^k \times \mathbb{R}^l)$-orbit of a subfamily (parametrized by $\mathbb{R}^{l-\operatorname{codim} P}$) of functionals $f_b$.

Finally, a short computation shows that the critical values of the functionals $f_b$ corresponding to the same fixed point $q \in Q$ are again of the form

$$f_b(u) = \langle \mu, \lambda \rangle - \int_0^1 h_t(\varphi_t(q))\, \mathrm{d}t \tag{40}$$

and hence they are arranged in a tower parametrized again by the set $\{\langle \mu, \eta \rangle \mid \eta \in \mathbb{Z}^k\}$.

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